Math 055 — Exam 1 Study Guide

This study guide is designed to help you prepare specifically for Exam 1. Every item below corresponds directly to skills or ideas found in the course material for first-order differential equations.

1. Autonomous Equations and Phase Portraits

What you should be able to do:

- Identify equilibrium solutions (critical points) by setting $f(y) = 0$.
- Sketch a phase portrait on the y-axis.
- Classify equilibria as asymptotically stable, unstable, or semi-stable.
- Predict long-term behavior [limit as $t -> \infty$] based on initial conditions.

You should understand:

- How the sign of $\frac{dy}{dt}$ determines whether $y(t)$ is increasing or decreasing.
- The relationship between the derivative $f'(y)$ at equilibrium and its stability.

Practice focus: Analyze $y' = y(y-3)(5-y)$. Find and classify all critical points.

2. Separable Differential Equations

What you should be able to do:

- Algebraically separate variables to the form $h(y)\,dy = g(x)\,dx$.
- Integrate both sides and solve for y (explicitly if requested, or implicitly).
- Apply initial conditions to find the particular constant C.

You should understand:

- Why lost solutions (singular solutions) might occur when dividing by functions of y.

Practice focus: Solve $\frac{dy}{dx} = \frac{x^2+1}{y^3}$ subject to $y(0) = 2$.

3. Linear Equations and Integrating Factors

What you should be able to do:

- Put a DE into standard form: $y' + P(x)y = Q(x)$.
- Calculate the integrating factor $\mu(x) = e^{\int P(x)dx}$.
- Solve for the general solution $y(x)$.

You should understand:

- How the product rule in reverse allows us to integrate the left side of the equation.

Practice focus: Solve $y' + 3y = 2e^x$ with $y(0) = 5$.

4. Exact Equations

What you should be able to do:
• Test for exactness using the partial derivative condition ($M_y = N_x$).
• Construct the potential function $f(x, y) = C$.
• Verify a solution by differentiating back to the original DE.

You should understand:
• The connection between exact DEs and conservative vector fields (gradient fields).

Practice focus: Solve $(3x^2y + 2x)dx + (x^3 + 1)dy = 0$.

5. Special Substitutions

What you should be able to do:
• Recognize homogeneous equations of degree n and use the substitution $v = y/x$.
• Recognize Bernoulli equations and use the substitution $u = y^{1-n}$.
• Convert the non-linear equation into a linear one.

You should understand:
• How changing variables can simplify the structure of a first-order equation.

Practice focus:
• Use a substitution to solve $y' = \frac{x^2+t^2}{xy}$.
• Solve the Bernoulli equation $y' + y = xy^3$.

6. Modeling with First-Order Equations

What you should be able to do:
• Construct a "Rate In - Rate Out" differential equation for mixing problems.
• Account for changing volumes when inflow and outflow rates differ.
• Identify carrying capacity and parameters in the logistic growth model.

You should understand:
• The physical meaning of each term in a growth or decay model.
• Long-term behavior (steady-state) in physical systems.

Practice focus: A tank holds 200L of water. Brine with 3kg/L salt enters at 4L/min and drains at 2L/min. Set up the IVP.

Practice Worksheet

1. Autonomous Equations

Consider $\frac{dy}{dt} = y^2(y - 5)$.

- Find all equilibrium solutions.
- Sketch the phase portrait.
- Classify each equilibrium as stable, unstable, or semi-stable.

2. Separable and Linear Equations

- Solve the IVP: $\frac{dy}{dx} = \frac{e^{2x}}{y^2}$, with $y(0) = 1$.
- Solve the IVP: $\frac{dy}{dx} - 3y = e^{3x}$, with $y(0) = 2$.

3. Exact Equations

Solve $(e^y + 2xy)dx + (xe^y + x^2 + 1)dy = 0$. Verify that your solution $f(x, y) = C$ satisfies the equation.

4. Substitution Methods

- Homogeneous: Solve $dy/dx = \frac{x^2+3y^2}{xy}$ using the substitution $v = y/x$.
- Bernoulli: Solve $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$ using an appropriate substitution.

5. Mixing Problems

A tank contains 500 liters of pure water. Brine containing 0.05 kg of salt per liter enters at 10 L/min. The mixture leaves at 10 L/min.

- Write the IVP for the amount of salt $A(t)$.
- Solve for $A(t)$ and determine the amount of salt in the tank after 30 minutes.

6. Conceptual Questions

- Explain the difference between a general solution and a particular solution.
- Describe the geometric meaning of an integrating factor in the context of non-exact equations.
- In a logistic growth model $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, what physical reality does M represent?